International Journal of Theoretical Physics, Vol. 29, No. 10, 1990

Exact Quantum Partition Function of the BCS Model

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Received May 2, 1990

The exact calculation of the reduced BCS model quantum partition function (QPF) in the thermodynamic limit is carried out by the path integration method. The expression for the QPF and the phase transition temperature T_c in the regular phase coincide with the results of Bogolyubov. In the nonregular phase a temperature singularity appears in the expression for the QPF: the QPF diverges in the region of temperatures T_c which are smaller than some critical temperature T_c^*, and it turns out that in all cases $T_c^* > T_c$ and the difference $T_c^* - T_c$ is not small. The interpretation of the temperature T_c^* is given.

Recently (Izmailov and Kessel, 1989*a–c*) the exact calculation of the reduced BCS model QPF was carried out by the path integration method over a Grassman manifold. These results revealed some interesting characteristics of the BCS model in the nonregular phase. There is another expression for the QPF, obtained in the thermodynamic limit by another method (Popov, 1981). One of the main purposes of this paper is to reveal the reasons for this difference. It turns out that the results of Popov (1981) can be obtained within our path integration approach when a definite basis GCS-2 of generalized coherent states (GCS) is used in order to construct the covariant symbol of the action operator. However, the GCS-2 basis does not satisfy some group-theoretic requirements due to the algebraic structure of the set of the operators from which the BCS model Hamiltonian is constructed. The use of another basis, GCS-1, devoid of this defect, leads to the results of Izmailov and Kessel (1989*a–c*) for the QPF. It is shown that the appeal to the perturbation theory does not allow us to make the choice between the different GCS bases. QPF is the S-matrix with the imaginary time parameter $i\tau$, $\tau \in [0, h\beta]$, and every term of the S-matrix perturbation theory (from the second one, which is proportional to the

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0020-7748/90/1000-1073$06.00/0

interaction constant squared) contains an arbitrariness in the form of additional integrals over arbitrary quasilocal operators. The source of this arbitrariness is in the uncertainty of the chronological operator product when the time variables coincide (Bogolyubov and Shirkov, 1959). So, due to this arbitrariness, it is possible to restore the results for the QPF obtained both in the GCS-1 and GCS-2 bases.

In this paper all calculations are carried out with a Hamiltonian that generalizes the Hamiltonian used in Izmailov and Kessel (1989*a*–*c*) in two respects. First, the constant of the four-fermion interaction potential is replaced by a separable potential of arbitrary type. Second, source terms of the Cooper pairs are added to the Hamiltonian of the BCS model (Bogolyubov, 1972). Thus, it is possible to obtain a united expression of the QPF for both the regular and the nonregular phases. In the limit cases, the QPF of our paper turns into the well-known result for the regular phase (Bogolyubov, 1972) and into our result (Izmailov and Kessel, 1989*a*–*c*) for the nonregular phase.

1. PATH INTEGRATION FORMALISM FOR THE REDUCED BCS MODEL

The quantum partition function of the system with the Hamiltonian H can be expressed in the thermodynamic limit with the help of path integration in the following way (Berezin, 1976; Klauder, 1979; Kuratsuji and Suzuki, 1980):

$$Z(\beta) \equiv \operatorname{Tr} e^{-\beta H} = Z \int d\mu \, e^{A_c/\hbar} \tag{1}$$

where the trace is taken over the states of the Hilbert space $\mathscr{H}$ which corresponds to the Hamiltonian H, $\beta = 1/kT$ (T is a temperature), Z is a normalization constant, and $d\mu$ is the Lebesgue measure invariant under the canonical transformations and defined on the space $\mathscr{H}_c$ of the covariant symbols of the operators which form the Hamiltonian H (the space $\mathscr{H}_c$ coincides with the classical phase space of the problem under consideration). The functional A_c, defined on space $\mathscr{H}_c$, is the covariant symbol of the action operator A. In general, A_c can be represented in the form

$$A_c = A_{c,\mathrm{kin}} - \int_0^{\hbar\beta} d\tau \, H_c(\tau) \tag{2}$$

Here $A_{c,\mathrm{kin}}$ is the kinematic part of the action operator covariant symbol. $H_c(\tau)$ is the covariant symbol of the Hamiltonian $H(\tau)$ in the Heisenberg representation. Generally speaking, the so-called extraintegral terms (Berezin, 1976; Vasil'ev, 1976) can appear side by side with the action operator covariant symbol A_c in the exponent index of the rhs of expression (1). However, they become zero if the integration in the space $\mathscr{H}_c$ is carried out over the closed trajectories (antiperiodic ones) that we shall deal with below. Let us note that sometimes it is more convenient to use the discrete representation for the integral over the variable τ (Berezin, 1976; Glimm and Jaffe, 1981; Vasil'ev, 1976):

$$\int_0^{\hbar\beta} d\tau f(\tau) = \lim_{N_\tau\to\infty} \frac{\hbar\beta}{N\tau} \sum_\tau f(\tau) \tag{3}$$

by dividing the interval of integration into N_τ indivisible equal parts. Thus, it is possible to present the spaces $\mathscr{H}$ and $\mathscr{H}_c$ in the form of direct products:

$$\mathscr{H} = \prod_\tau \otimes \mathscr{H}(\tau), \qquad \mathscr{H}_c = \prod_\tau \otimes \mathscr{H}_c(\tau) \tag{4}$$

The invariant integration measure $d\mu$ can be represented in the analogous form

$$d\mu = \prod_\tau \otimes d\mu(\tau) \tag{5}$$

In the problem under consideration the Hamiltonian H is the Hamiltonian of the well-known reduced BCS model (Bogolyubov, 1972):

$$\begin{aligned} H &= H_0 + V_1 + V_2 \\ H_0 &= \sum_{\mathbf{k}} \sum_{\sigma} t_{\mathbf{k}} n_\sigma(\mathbf{k}, 0) \\ n_\sigma(\mathbf{k}, 0) &= \Psi_\sigma^+(\mathbf{k}, 0)\Psi_\sigma(\mathbf{k}, 0) \\ V_1 &= -\Delta \sum_{\mathbf{k}_1} \lambda(\mathbf{k})[\Delta(\mathbf{k}, 0) + \Delta^+(\mathbf{k}, 0)] \\ \Delta(\mathbf{k}, 0) &= \Psi_{-1/2}(-\mathbf{k}, 0)\Psi_{1/2}(\mathbf{k}, 0) \\ V_2 &= -\frac{U}{N} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \lambda(\mathbf{k}_1)\lambda(\mathbf{k}_2)\Delta^+(\mathbf{k}_1, 0)\Delta(\mathbf{k}_2, 0) \end{aligned} \tag{6}$$

where $\Psi_\sigma^+(\mathbf{k}, \tau)$ and $\Psi_\sigma(\mathbf{k}, \tau)$ are the Fermi creation and annihilation field operators of the conduction electron with momentum $\hbar\mathbf{k}$, spin projection on the quantization axis $\sigma = \pm 1/2$, and kinetic energy $t_{\mathbf{k}} = (\hbar\mathbf{k})^2/2M - t_F$, which is counted from the Fermi energy t_F at the time moment τ; M is the electron mass; N is the number of elementary cells in the crystal; Δ is some positive energetic constant, which has the sense of the gap in the electron spectrum when these electrons are in the superconducting phase; U is the constant s-wave part of the electron–electron interaction potential; $\lambda(k)$ is the form factor, which possesses the following property: $\lambda(-\mathbf{k}) = -\lambda(\mathbf{k})$. Thus, the Hamiltonian (6) generalizes the energy operator investigated

(Izmailov and Kessel, 1989a–c) and turns into it when $\Delta = 0$ and $\lambda(\mathbf{k}) = 1$.[2] The operator V_2 is the source of the Cooper pairs and its presence in the total Hamiltonian H violates the symmetry of the physical system under consideration: the operator H loses its global gauge invariance and the particle number is no longer a constant of motion. The introduction of the additional quantum number $\hbar\mathbf{k}$ makes it possible to construct the following more detailed direct products for the spaces $\mathcal{H}(\tau)$ and $\mathcal{H}_c(\tau)$,

$$\mathcal{H}(\tau) = \prod_{\mathbf{k}} \otimes \mathcal{H}(\mathbf{k}, \tau), \qquad \mathcal{H}_c(\tau) = \prod_{\mathbf{k}} \otimes \mathcal{H}_c(\mathbf{k}, \tau) \tag{7}$$

and for the measure $d\mu(\tau)$,

$$d\mu(\tau) = \prod_{\mathbf{k}} \otimes d\mu(\mathbf{k}, \tau) \tag{8}$$

In order to derive the reduced BCS model action functional A_c it is necessary to find the covariant symbols $\Psi^*_{c,\sigma}(\mathbf{k}, \tau)$ and $\Psi_{c,\sigma}(\mathbf{k}, \tau)$ of the field operators $\Psi^+_\sigma(\mathbf{k}, \tau)$ and $\Psi_\sigma(\mathbf{k}, \tau)$. By definition, these symbols are the diagonal matrix elements of the corresponding operators in the basis of the space $\mathcal{H}(\mathbf{k}, \tau)$ generalized coherent states $-|\xi, \Psi_c(\mathbf{k}, \tau)\rangle$, where ξ is a complex numerical parameter. The Fermi field GCS possess the following properties. They are the eigenfunctions of the field annihilation operators:

$$\Psi_\sigma(\mathbf{k}, \tau)|\xi; \Psi_c(\mathbf{k}, \tau)\rangle = \Psi_{c,\sigma}(\mathbf{k}, \tau)|\xi; \Psi_c(\mathbf{k}, \tau)\rangle \tag{9}$$

The set of these functions is overcomplete and thus it is not orthonormal:

$$\int d\mu(\mathbf{k}, \tau)\langle\Psi_c(\mathbf{k}, \tau); \eta|\xi; \Psi_c(\mathbf{k}, \tau)\rangle = F(\xi - \eta) \neq 0 \tag{10}$$

where $F(0) = 1$. Moreover, for $\xi = \xi_f$ the following resolution of unity must exist:

$$\int d\mu(\mathbf{k}, \tau)|\xi_f; \Psi_c(\mathbf{k}, \tau)\rangle\langle\Psi_c(\mathbf{k}, \tau); \xi_f| = \mathbb{1}(\mathbf{k}, \tau) \tag{11}$$

where $\mathbb{1}(\mathbf{k}, \tau)$ is the unity operator, which acts in the space $\mathcal{H}_c(\mathbf{k}, \tau)$. The integrals in expressions (10) and (11) are functional ones over the space of functions which take their values in the Grassmann algebra. The invariant Lebesque measure of such an integration is chosen in the form (Berezin, 1976; Ohnuki and Kashiwa, 1978; Vasil'ev, 1976)

$$d\mu(\mathbf{k}, \tau) = \prod_\sigma d\Psi^*_{c,\sigma}(\mathbf{k}, \tau)\, d\Psi_{c,\sigma}(\mathbf{k}, \tau) \tag{12}$$

Usually, in accordance with these requirements, the GCS for the fermion system are taken in the following form (Ohnuki and Kashiwa,

[2]The energy operator considered in Izmailov and Kessel (1989a–c) completely coincides with the Hamiltonian of the Thirring model (Thirring, 1968).

1978); Perelomov, 1986):

$$|\xi, \Psi_c(\mathbf{k}, \tau)\rangle_2 = T_{\xi,2}(\mathbf{k}, \tau)|0(\mathbf{k}, \tau)\rangle \tag{13}$$

$$T_{\xi,2}(\mathbf{k}, \tau) = \exp\{-\xi[\Psi^*_{c,1/2}(\mathbf{k}, \tau)\Psi^+_{1/2}(\mathbf{k}, \tau) + \Psi^*_{c,-1/2}(-\mathbf{k}, \tau)\Psi^+_{-1/2}(-\mathbf{k}, \tau)]\}$$

where $|0(\mathbf{k}, \tau)\rangle$ is the eigenvector of the field annihilation operators [the vacuum vector of the space $\mathscr{H}(\mathbf{k}, \tau)$], which corresponds to the zero eigenvalue.

However, the GCS-2 of the fermion system taken in the form (13) are not quite satisfactory from the group theory point of view. The fact is that the operators $T_\xi(k, \tau)$ must form an irreducible representation of the factor algebra $\bar{\mathscr{A}} = \mathscr{A}/\mathscr{A}_s$ of the model under consideration. For the reduced BCS model, the total Lie superalgebra of the operators from which the model Hamiltonian H is constructed is

$$\mathscr{A} = \prod_{\mathbf{k}} \prod_{\sigma} \otimes \{\mathbb{1}(\pm\mathbf{k}, \tau); \Psi^+_{1/2}(\mathbf{k}, \tau); \Psi_{1/2}(\mathbf{k}, \tau); \Psi^+_{-1/2}(-\mathbf{k}, \tau); \Psi_{-1/2}(-\mathbf{k}, \tau);$$
$$n_{1/2}(\mathbf{k}, \tau); n_{-1/2}(-\mathbf{k}, \tau); \Delta(\mathbf{k}, \tau); \Delta^+(\mathbf{k}, \tau)\} \tag{14}$$

and the vacuum state $|0\rangle = \prod_{\mathbf{k}} \prod_\tau \otimes |0(\mathbf{k}, \tau)\rangle$ stability Lie superalgebra $\mathscr{A}_s$ ($\mathscr{A}_s|0\rangle = \lambda|0\rangle$) consists of the following basis elements:

$$\mathscr{A}_s = \prod_{\mathbf{k}} \prod_{\sigma} \otimes \{\mathbb{1}(\pm\mathbf{k}, \tau); \Psi_{1/2}(\mathbf{k}, \tau); \Psi_{-1/2}(-\mathbf{k}, \tau);$$
$$n_{1/2}(\mathbf{k}, \tau); n_{-1/2}(-\mathbf{k}, \tau); \Delta(\mathbf{k}, \tau)\} \tag{15}$$

Thus, the factor algebra $\bar{\mathscr{A}}$ of the model under consideration is

$$\bar{\mathscr{A}} = \prod_{\mathbf{k}} \prod_{\tau} \otimes \{\Psi^+_{1/2}(\mathbf{k}, \tau); \Psi^+_{-1/2}(-\mathbf{k}, \tau); \Delta^+(\mathbf{k}, \tau)\} \tag{16}$$

and hence, in contrast to expression (13), the irreducible representation operator $T_\xi(\mathbf{k}, \tau)$ must be expressed with the help of the operator $\Delta^+(\mathbf{k}, \tau)$, too. Taking into account all the above properties of the fermion coherent states (9)-(11), one must choose the irreducible representation operator $T_\xi(\mathbf{k}, \tau)$ in the form

$$\begin{aligned} T_{\xi,1}(\mathbf{k}, \tau) = Z_\xi(\mathbf{k}, \tau)\{1 - \xi[&\Psi^*_{c,1/2}(\mathbf{k}, \tau)\Psi^+_{1/2}(\mathbf{k}, \tau) \\ &+ \Psi^*_{c,-1/2}(-\mathbf{k}, \tau)\Psi^+_{-1/2}(-\mathbf{k}, \tau)] \\ &+ \frac{\xi^2}{\sqrt{2}}[\Delta_c(\mathbf{k}, \tau) + \Delta^*_c(\mathbf{k}, \tau)]\Delta^+(\mathbf{k}, \tau)\} \end{aligned} \tag{17}$$

$$\begin{aligned} Z_\xi(\mathbf{k}, \tau) = 1 &- \frac{|\xi|^2}{2}[n_{c,1/2}(\mathbf{k}, \tau) + n_{c,-1/2}(-\mathbf{k}, \tau)] \\ &+ \frac{|\xi|^4}{4} n_{c,1/2}(\mathbf{k}, \tau) n_{c,-1/2}(-\mathbf{k}, \tau) \end{aligned}$$

$$n_{c,\sigma}(\mathbf{k}, \tau) = \Psi^*_{c,\sigma}(\mathbf{k}, \tau)\Psi_{c,\sigma}(\mathbf{k}, \tau)$$

$$\Delta_c(\mathbf{k}, \tau) = \Psi_{c,-1/2}(-\mathbf{k}, \tau)\Psi_{c,1/2}(\mathbf{k}, \tau)$$

where the implementation of property (11) is secured by the condition $\xi = \xi_f = \exp(\pm i\pi/8)$. In the GCS-1 basis, which is defined by the relation

$$|\xi, \Psi_c(\mathbf{k}, \tau)\rangle_1 = T_{\xi,1}(\mathbf{k}, \tau)|0(\mathbf{k}, \tau)\rangle \tag{18}$$

the covariant symbol A_c of the reduced BCS model action operator takes the following form:

$$\frac{A_c}{\hbar} = \lim_{N_\tau \to \infty} \frac{1}{N_\tau} \sum_\tau \frac{A_c(\tau)}{\hbar} \tag{19}$$

$$\begin{aligned}
\frac{A_c(\tau)}{\hbar} = \sum_{\mathbf{k}} \{&\Psi^*_{c,1/2}(\mathbf{k}, \tau)\, W_{1/2}(\mathbf{k}, \tau)\Psi_{c,1/2}(\mathbf{k}, \tau)\\
&+ \Psi^*_{c,-1/2}(-\mathbf{k}, \tau)\, W(-\mathbf{k}, \tau)\Psi_{c,-1/2}(-\mathbf{k}, \tau)\\
&+ \Delta\beta\lambda(\mathbf{k})[\Delta_c(\mathbf{k}, \tau) + \Delta^*_c(\mathbf{k}, \tau)]\}\\
&+ \frac{U\beta}{2N} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \lambda(\mathbf{k}_1)\lambda(\mathbf{k}_2)[\Delta_c(\mathbf{k}_1, \tau) + \Delta^*_c(\mathbf{k}_1, \tau)]\\
&\times [\Delta_c(\mathbf{k}_2, \tau) + \Delta^*_c(\mathbf{k}_2, \tau)]
\end{aligned}$$

$$W(\mathbf{k}, \tau) = \hbar\beta\partial\tau - \beta t_{\mathbf{k}}$$

where symbol ∂_τ designates the continuous derivative in the interval $[(\tau - 1)\hbar\beta/N\tau, \tau\hbar\beta/N_\tau]$.

2. CALCULATION OF THE REDUCED BCS MODEL QUANTUM PARTITION FUNCTION

In the case when the action covariant symbol A_c has the form (19), expression (1) for the reduced BCS model quantum partition function in the thermodynamic limit can be represented in the following way:

$$Z_1(\beta) = \mathrm{Sp}\; e^{-\beta H} = Z \lim_{N \to \infty} \lim_{N_\tau \to \infty} \lim_{\varepsilon \to 0} \det{}_{(1)} \hat{B}(\varepsilon) \tag{20}$$

$$\hat{B}(\varepsilon) = \sum_\tau \sum_{\tau(\varepsilon)} B(\varepsilon, \tau)\hat{p}(\tau, \tau(\varepsilon))$$

$$B(\varepsilon, \tau) = \langle\tau|B(\tau, \tau(\varepsilon))|\tau(\varepsilon)\rangle$$

where $\lim_{\varepsilon \to 0} \tau(\varepsilon) = \tau$ and $B(\varepsilon, \tau)$ is the kernel of the differentiation operator:

$$B(\tau, \tau(\varepsilon)) = \int d\mu_{1/2}(\tau)\, d\mu_{-1/2}(\tau(\varepsilon)) \exp\left[\frac{A_c(\tau, \tau(\varepsilon))}{\hbar}\right] \tag{21}$$

$$\frac{A_c(\tau, \tau(\varepsilon))}{\hbar} = \sum_{\mathbf{k}} \{\Psi^*_{c,1/2}(\mathbf{k}, \tau)\, W(\mathbf{k}, \tau)\Psi_{c,1/2}(\mathbf{k}, \tau)$$
$$+\Psi^*_{c,-1/2}(-\mathbf{k}, \tau(\varepsilon))\, W(-\mathbf{k}, \tau(\varepsilon))\Psi_{c,-1/2}(-\mathbf{k}, \tau(\varepsilon))$$
$$+\Delta\beta\lambda(\mathbf{k})[\Delta_c(\mathbf{k}; \tau, \tau(\varepsilon)) + \Delta^*_c(\mathbf{k}; \tau, \tau(\varepsilon))]\}$$
$$-\frac{U\beta}{2N}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2} \lambda(\mathbf{k}_1)\lambda(\mathbf{k}_2)[\Delta_c(\mathbf{k}_1; \tau, \tau(\varepsilon)) + \Delta^*_c(\mathbf{k}_1; \tau, \tau(\varepsilon))]$$
$$\times[\Delta_c(\mathbf{k}_2; \tau, \tau(\varepsilon)) + \Delta^*_c(\mathbf{k}_2; \tau, \tau(\varepsilon))] \tag{22}$$
$$\Delta_c(\mathbf{k}; \tau, \tau(\varepsilon)) = \Psi_{c,-1/2}(-\mathbf{k}, \tau)\Psi_{c,1/2}(\mathbf{k}, \tau)$$

The set of the projection operators $\{\hat{p}(\tau, \tau')\}$ entering in expression (20) forms the Okubo basis in the space of the square matrix algebra $\mathrm{Mat}(N_\tau, \mathbb{C})$ and possesses the property

$$\hat{p}(\tau, \tau')\hat{p}(\gamma, \gamma') = \delta_{\tau;\gamma}\hat{p}(\tau, \gamma') \tag{23}$$

Here $\det_{(1)} \hat{X}$ is the notation for the matrix $\hat{X}$ determinant in the basis $\{\hat{p}(\tau, \tau')\}$. Bra $\langle\tau|$ and ket $|\tau\rangle$ vectors of this basis can be represented in the form

$$\langle\tau| = \frac{1}{\sqrt{N\tau}}\sum_{\omega} e^{i\omega\tau}\hat{p}_{\omega_f\omega}, \qquad |\tau\rangle = \frac{1}{\sqrt{N\tau}}\sum_{\omega} e^{-i\omega\tau}\hat{p}_{\omega\omega_f} \tag{24}$$

where $\omega = \pi(2n+1)/\hbar\beta$ $(n = 0, \pm 1, \pm 2, \ldots, \pm N\tau)$ for the problems in which the path integration is carried out over the Grassmann variables; ω_f is some fixed frequency, from which all the other frequencies are counted; $\{\hat{p}_{\omega\omega'}\}$ is the dual Okubo basis, which is the Fourier image of the basis $\{\hat{p}(\tau, \tau')\}$:

$$\hat{p}(\tau, \tau') = \frac{1}{N_\tau}\sum_{\Omega}\sum_{\Omega'} e^{i(\Omega\tau - \Omega'\tau')}\,\hat{p}_{\Omega\Omega'} \tag{25}$$

where the frequencies Ω can take either the values $\omega = \pi(2n+1)/\hbar\beta$ or the values $\nu = 2\pi n/\hbar\beta$.

Let us rewrite the expression for $\exp[A_c(\tau, \tau(\varepsilon))/\hbar]$ in the following way:

$$\exp\left[\frac{A_c(\tau, \tau(\varepsilon))}{\hbar}\right] = \exp\left[\frac{A_{c,1}(\tau, \tau(\varepsilon))}{\hbar}\right]\langle\exp[A_{c,2}(\tau, \tau(\varepsilon); \varphi(\tau))]\rangle_{0,\varphi(\tau)} \tag{26}$$

$$\frac{A_{c,1}(\tau, \tau(\varepsilon))}{\hbar} = \frac{1}{N_\tau}\sum_{\mathbf{k}} \{\Psi^*_{c,1/2}(\mathbf{k}, \tau)\, W(\mathbf{k}, \tau)\Psi_{c,1/2}(\mathbf{k}, \tau)$$
$$+\Psi^*_{c,-1/2}(-\mathbf{k}, \tau(\varepsilon))\, W(-\mathbf{k}, \tau(\varepsilon))\Psi_{c,-1/2}(-\mathbf{k}, \tau(\varepsilon))$$
$$+\Delta\beta\lambda(\mathbf{k})[\Delta_c(\mathbf{k}; \tau, \tau(\varepsilon)) + \Delta^*_c(\mathbf{k}; \tau, \tau(\varepsilon))]\}$$

$$\frac{A_{c,2}(\tau, \tau(\varepsilon); \varphi(\tau))}{\hbar} = i\varphi(\tau)\sqrt{\frac{U\beta}{NN_\tau}}\sum_{\mathbf{k}} \lambda(\mathbf{k})[\Delta_c(\mathbf{k}; \tau, \tau(\varepsilon)) + \Delta_c^*(\mathbf{k}; \tau, \tau(\varepsilon))]$$

Here $\langle \cdots \rangle_{0,\varphi(\tau)}$ is the average over the vacuum of some Bose field $\varphi(\tau)$, which possesses the following properties:

$$\varphi(\tau) = \varphi^-(\tau) + \varphi^+(\tau), \qquad [\varphi^-(\tau), \varphi^+(\tau')] = \delta_{\tau,\tau'} \tag{27}$$

For the subsequent calculations it will be necessary to know the expression for the vacuum average value of the operator $\varphi^n(\tau)$:

$$\langle \varphi^n(\tau) \rangle_{0,\varphi(\tau)} = \delta_{n,2m} \frac{(2m)!}{2^m m!} \tag{28}$$

Now there are only quadratic forms of the Fermi operator covariant symbols in the exponent index of expression (26).

Making the direct calculation of the path integral (21), one can obtain the explicit expression for the differentiation operator $B(\tau, \tau(\varepsilon))$:

$$\begin{aligned} B(\tau, \tau(\varepsilon)) = \sum_{m=0}^{N} \sum_{n=0}^{N-m} & \frac{(2m+2n)!\,(\Delta\beta)^{2n}(U\beta N_\tau/2N)^m}{(2n)!\,m!\,[(m+n)!\,(N-m-n)!]^2} \\ & \times \sum_{\mathbf{k}_1\mathbf{k}_1'} \cdots \sum_{\mathbf{k}_{m+n}\mathbf{k}_{m+n}'} \sum_{\mathbf{p}_1\mathbf{q}_1} \cdots \sum_{\mathbf{p}_{N-m-n}\mathbf{q}_{N-m-n}} \\ & \times \varepsilon^{\mathbf{k}_1\cdots\mathbf{k}_{m+n}\mathbf{p}_1\cdots\mathbf{p}_{N-m-n}}_{\mathbf{k}_1'\cdots\mathbf{k}_{m+n}'\mathbf{p}_1\cdots\mathbf{p}_{N-m-n}} \varepsilon^{\mathbf{k}_1\cdots\mathbf{k}_{m+n}\mathbf{q}_1\cdots\mathbf{q}_{N-m-n}}_{\mathbf{k}_1'\cdots\mathbf{k}_{m+n}'\mathbf{q}_1\cdots\mathbf{q}_{N-m-n}} \\ & \times \prod_{j=1}^{m+n} \lambda(\mathbf{k}_j)\lambda(\mathbf{k}_j') \prod_{c=1}^{N-m-n} W(\mathbf{p}_e, \tau)\, W(\mathbf{q}_e, \tau(\varepsilon)) \end{aligned} \tag{29}$$

where

$$\varepsilon^{\mathbf{k}_1\cdots\mathbf{k}_N}_{\mathbf{k}_1'\cdots\mathbf{k}_N'}$$

is the generalized Kronecker tensor of rank N. Result (28) was used in order to obtain expression (29). Let us take into account now the properties of the generalized Kronecker tensors:

$$\sum_{\mathbf{k}_{p+1}} \cdots \sum_{\mathbf{k}_N} \varepsilon^{\mathbf{k}_1\cdots\mathbf{k}_p\mathbf{k}_{p+1}\cdots\mathbf{k}_N}_{\mathbf{k}_1'\cdots\mathbf{k}_p'\mathbf{k}_{p+1}\cdots\mathbf{k}_N} = (N-p)!\,\varepsilon^{\mathbf{k}_1\cdots\mathbf{k}_p}_{\mathbf{k}_1'\cdots\mathbf{k}_p'} \tag{30}$$

$$\sum_{\mathbf{k}_1'} \cdots \sum_{\mathbf{k}_p'} \varepsilon^{\mathbf{k}_1\cdots\mathbf{k}_p}_{\mathbf{k}_1'\cdots\mathbf{k}_p'} \varepsilon^{\mathbf{k}_1\cdots\mathbf{k}_p}_{\mathbf{k}_1'\cdots\mathbf{k}_p'} \prod_{j=1}^{p} Z_{\mathbf{k}_j'} = p!\,\varepsilon^{\mathbf{k}_1\cdots\mathbf{k}_p}_{\mathbf{k}_1\cdots\mathbf{k}_p} \prod_{j=1}^{p} Z_{\mathbf{k}_j} \tag{31}$$

where $Z_{\mathbf{k}}$ is the ordinary non-Grassmann function of variable $\mathbf{k}$ and $p = m+n$. Thus, the operator $B(\tau, \tau(\varepsilon))$ can be represented in the form

$$B(\tau, \tau(\varepsilon)) = D(\tau, \tau(\varepsilon)) F(\tau, \tau(\varepsilon)) \tag{32}$$

$$D(\tau, \tau(\varepsilon)) = \prod_{\mathbf{k}} W(\mathbf{k}, \tau) W(-\mathbf{k}, \tau(\varepsilon))$$

$$F(\tau, \tau(\varepsilon)) = \sum_{m=0}^{N} \sum_{n=0}^{N-m} \frac{(2m+2n)!\,(\Delta\beta)^{2n}(U\beta N\tau/N)^m}{(2n)!\,m!\,(m+n)!}$$

$$\times \sum_{\mathbf{k}_1} \cdots \sum_{\mathbf{k}_n} \varepsilon^{\mathbf{k}_1 \cdots \mathbf{k}_n}_{\mathbf{k}_1 \cdots \mathbf{k}_n} \prod_{j=1}^{n} V(\mathbf{k}_j; \tau, \tau(\varepsilon))$$

$$\times \sum_{\substack{\mathbf{k}_{n+1} \\ (\mathbf{k}_1,\ldots,\mathbf{k}_n \neq \mathbf{k}_{n+1},\ldots,\mathbf{k}_{n+m})}} \cdots \sum_{\mathbf{k}_{n+m}} \varepsilon^{\mathbf{k}_{n+1} \cdots \mathbf{k}_{n+m}}_{\mathbf{k}_{n+1} \cdots \mathbf{k}_{n+m}} \prod_{l=1}^{n+m} V(\mathbf{k}_l; \tau, \tau(\varepsilon))$$

where

$$V(\mathbf{k}; \tau, \tau(\varepsilon)) = \lambda^2(\mathbf{k}) W^{-1}(\mathbf{k}, \tau) W^{-1}(-\mathbf{k}, \tau(\varepsilon))$$

The introduction of the auxiliary differentiation operator

$$G(b_1, b_2) = \sum_{l_1=0}^{N} \sum_{l_2=0}^{N} \frac{(2l_1+2l_2)!}{(2l_1)!\,(2l_2)!\,(l_1+l_2)!} \frac{d^{l_1}}{db_1^{l_1}} \frac{d^{l_2}}{db_2^{l_2}} \tag{33}$$

makes it possible to obtain in the thermodynamic limit the following expression for the pseudodifferentiation operator $F(\tau, \tau(\varepsilon))$:

$$F(\tau, \tau(\varepsilon)) = \lim_{N\to\infty} \lim_{b_1\to 0} \lim_{b_2 \to 0} G(b_1, b_2) \sum_{m=0}^{N} \sum_{n=0}^{N-m} \frac{[b_1(\Delta\beta)^2]^n}{n!} \sum_{\mathbf{k}_1} \cdots \sum_{\mathbf{k}_n} \varepsilon^{\mathbf{k}_1 \cdots \mathbf{k}_n}_{\mathbf{k}_1 \cdots \mathbf{k}_n}$$

$$\times \prod_{j=1}^{n} V(\mathbf{k}_j; \tau, \tau(\varepsilon)) \frac{(2m)!}{(m!)^2} \left[\frac{b_2 U\beta N_\tau}{N} \sum_{\substack{\mathbf{k} \\ (\mathbf{k} \neq \mathbf{k}_1,\ldots,\mathbf{k}_n)}} V(\mathbf{k}; \tau, \tau(\varepsilon)) \right]^m \tag{34}$$

Having used the representation

$$\left[\frac{b_2 U\beta N_\tau}{N} \sum_{\substack{\mathbf{k} \\ (\mathbf{k} \neq \mathbf{k}_1,\ldots,\mathbf{k}_n)}} (\mathbf{k}; \tau, \tau(\varepsilon)) \right]^m$$

$$= \prod_{j=1}^{n} \lim_{a_{\mathbf{k}_j} \to 0} e^{-d/da_{\mathbf{k}_j}} \left[\frac{b_2 U\beta N_\tau}{N} \sum_{\mathbf{k}} (1 + a_{\mathbf{k}}) V(\mathbf{k}; \tau, \tau(\varepsilon)) \right]^m \tag{35}$$

one can derive in the thermodynamic limit the operator $B(\tau, \tau(\varepsilon))$ in the form:

$$B(\tau, \tau(\varepsilon)) = \lim_{N\to\infty} \lim_{b_1\to 0} \lim_{b_2\to 0} G(b_1, b_2)$$
$$\times \prod_{\mathbf{k}} \lim_{a_{\mathbf{k}}\to 0} \{ W(\mathbf{k}, \tau)\, W(-\mathbf{k}, \tau(\varepsilon)) + b_1[\Delta\beta\lambda(\mathbf{k})]^2\, e^{-d/da_{\mathbf{k}}} \}$$
$$\times \left[1 - b_2 \frac{2U\beta N_\tau}{N} \sum_{\mathbf{k}} (1 + a_{\mathbf{k}})\, V(\mathbf{k};\, \tau, \tau(\varepsilon)) \right]^{-1/2} \tag{36}$$

In order to derive expression (36), the following summation formula was used:

$$\sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} X^m = (1-4X)^{-1/2}, \qquad X \in [-\tfrac{1}{4}, \tfrac{1}{4}) \tag{37}$$

It results from this that in our problem the QPF converges under the condition

$$b_2 \frac{2U\beta N_\tau}{N} \sum_{\mathbf{k}} (1 + a_{\mathbf{k}}) \langle \tau | V(\mathbf{k};\, \tau, \tau(\varepsilon)) | \tau(\varepsilon) \rangle \in [-1, 1) \tag{38}$$

Insofar as it was implied above that the parameter $b_2 \to 0$, this condition is practically satisfied in all cases. Taking into account that the introduction of this parameter was an auxiliary procedure and that the fulfilment of the same condition with $b_2 = 1$ will be demanded in Section 3, it is natural to demand the fulfillment of condition (38) in the more rigid form which corresponds to $b_2 = 1$.

In the following stage of our calculations it is necessary to find the determinant (20) of the matrix $\hat{B}(\varepsilon)$. For this purpose one has to take into account that the operator $V(\mathbf{k};\, \tau, \tau(\varepsilon))$ entering into expression (36) for the $B(\tau, \tau(\varepsilon))$ is the pseudodifferential operator and the operator $W(\mathbf{k}, \tau)\, W(-\mathbf{k}, \tau(\varepsilon))$ is the differential (the so-called eigenpseudodifferential) one (Treves, 1982). The matrix elements of the above operators which are used for the calculation of the determinant (20) can be represented in the form

$$\langle \tau | W(\mathbf{k}, \tau)\, W(-\mathbf{k}, \tau(\varepsilon)) | \tau(\varepsilon) \rangle$$
$$= W(\mathbf{k}, \tau)\, W(-\mathbf{k}, \tau(\varepsilon))\, \delta_{\tau, \tau(\varepsilon)} = \sum_{\omega} F_1(\omega)\, e^{i\omega(\tau - \tau(\varepsilon))} \tag{39}$$

$$\langle \tau | W^{-1}(\mathbf{k}, \tau)\, W^{-1}(-\mathbf{k}, \tau(\varepsilon)) | \tau(\varepsilon) \rangle$$
$$= W^{-1}(\mathbf{k}, \tau)\, W^{-1}(-\mathbf{k}, \tau(\varepsilon))\, \delta_{\tau, \tau(\varepsilon)} = \sum_{\nu} F_2(\nu)\, e^{i\nu(\tau - \tau(\varepsilon))} \tag{40}$$

In the theory of Fourier integral operators, which deals with pseudodifferential operators, it is shown that the symbol $F_1(\omega)$ of the differential operator $W(\mathbf{k}, \tau)\, W(-\mathbf{k}, \tau(\varepsilon))$ can be found with the help of the formula

$$F_1(\omega) = \lim_{\varepsilon\to 0} W(\mathbf{k}, \tau)\, W(-\mathbf{k}, \tau(\varepsilon))\, e^{i\omega(\tau-\tau(\varepsilon))} = (\hbar\beta\omega)^2 + (\beta t_{\mathbf{k}})^2 \tag{41}$$

and the symbol $F_2(\nu)$ of the pseudodifferential operator

$$W^{-1}(\mathbf{k}, \tau)\, W^{-1}(-\mathbf{k}, \tau(\varepsilon))$$

can be found with the help of the formula

$$F_2(\nu) = N_\tau \lim_{\varepsilon\to 0} e^{-\nu\tau} W^{-1}(\mathbf{k}, \tau)\, W^{-1}(-\mathbf{k}, \tau(\varepsilon))\, e^{i\nu\tau}\delta_{\tau,\tau(\varepsilon)}$$

$$= \sum_\omega \frac{1}{[i\hbar\beta(\nu-\omega) - \beta t_{\mathbf{k}}](i\hbar\beta\omega - \beta t_{\mathbf{k}})} \tag{42}$$

In order to derive expression (42), it was taken into account that the pseudodifferential operator $W^{-1}(\mathbf{k}, \tau)\, W^{-1}(-\mathbf{k}, \tau(\varepsilon))$ can be considered as a differential operator on the compact open subset Λ, where $\delta_{\tau,\tau(\varepsilon)}(\Lambda) = 1$ (Treves, 1982).

Substituting now expressions (39)-(40) into (36) and then into (20), it is possible to obtain the following expression for the QPF of the reduced BCS model:

$$\begin{aligned} Z_1(\beta) = Z &\lim_{N\to\infty} \prod_\omega \lim_{b_1\to 0} \lim_{b_2\to 0} G(b_1, b_2) \\ &\times \prod_{\mathbf{k}} \lim_{a_{\mathbf{k}}\to 0} \{(\hbar\beta\omega)^2 + (\beta t_{\mathbf{k}})^2 + b_1[\Delta\beta\lambda(\mathbf{k})]^2\, e^{-d/da_{\mathbf{k}}}\} \\ &\times \left[1 - \mathrm{b}_2 \frac{2\mathrm{U}\beta}{\mathrm{N}} \sum_{\mathbf{k}} (1+a_{\mathbf{k}}) R\left(\mathbf{k}, \omega - \frac{\pi}{\hbar\beta}\right)\right]^{-1/2} \end{aligned} \tag{43}$$

$$\begin{aligned} R\left(\mathbf{k}, \omega - \frac{\pi}{\hbar\beta}\right) &\equiv R(\mathbf{k}, \nu) \\ &= \lambda^2(\mathbf{k}) \sum_{\omega_1} \frac{1}{[i\hbar\beta(\nu-\omega_1) - \beta t_{\mathbf{k}}](i\hbar\beta\omega_1 - \beta t_{\mathbf{k}})} \\ &= -\lambda^2(\mathbf{k}) \frac{th(\beta t_{\mathbf{k}}/2)}{i\hbar\beta\nu - 2\beta t_{\mathbf{k}}} \end{aligned}$$

In order to obtain the expression for the $R(k, \nu)$ the following well-known formula was used:

$$\sum_\omega \frac{1}{i\hbar\beta\omega - \beta a} = -\tfrac{1}{2}\, th\left(\frac{\beta a}{2}\right) \tag{44}$$

In the dual space of the Okubo basis, condition (38) takes the form

$$b_2 \frac{2U\beta}{N} \sum_{\mathbf{k}} (1+a_{\mathbf{k}})\lambda^2(\mathbf{k}) \frac{th(\beta t_{\mathbf{k}}/2)}{i\hbar\beta\nu - 2\beta t_{\mathbf{k}}} \in (-1, 1] \tag{45}$$

Expression (39) is the final result of our exact calculation in the thermodynamic limit of the QPF for the reduced BCS model with the Hamiltonian in the form (6).

3. SOME PROPERTIES OF THE REDUCED BCS MODEL QUANTUM PARTITION FUNCTION

Let us carry out the analysis of expression (43) for the QPF in two limiting cases.

1. $U=0$. In this case the Hamiltonian (6) turns into the well-known approximation Hamiltonian of the BCS model, which is used for the description of the model in the regular phase, and expression (43) becomes the well-known exact result (Bogolyubov, 1972) for the QPF of the system with the Hamiltonian $H_0 + V_1$:

$$Z_1(\beta)|_{U=0} = \text{Sp}\, e^{-\beta(H_0+V_1)} = Z \prod_{\omega} \prod_{\mathbf{k}} \{(\hbar\beta\omega)^2 + (\beta t_{\mathbf{k}})^2 + [\Delta\beta\lambda(\mathbf{k})]^2\} \tag{46}$$

The estimation of the critical temperature in the regular phase, made within the traditional assumption (Bogolyubov, 1972), is given by the expression

$$T_c = T_{\mathrm{D}}\, e^{-1/\lambda}, \qquad T_{\mathrm{D}} = \frac{2\gamma\hbar\omega_{\mathrm{D}}}{\pi k}, \qquad \lambda = \frac{\rho U}{N} \tag{47}$$

where γ is the Euler constant, ω_{D} is the Debye frequency for the definite crystal, and ρ is the density of the conduction electron states on the Fermi surface.

2. $\Delta=0$. In this case expression (39) becomes the result obtained (Izmailov and Kessel, 1989*a–c*) for the reduced QPF in the nonregular phase and in the thermodynamic limit:

$$Z_1(\beta)|_{\Delta=0} = \text{Sp}\, e^{-\beta(H_0+V_2)} \Rightarrow \lim_{N\to\infty} \text{Sp}\, e^{-\beta H_0} F_1(\beta) \tag{48}$$

$$\text{Sp}\, e^{-\beta H_0} = Z \prod_{\omega} \prod_{\mathbf{k}} [(\hbar\beta\omega)^2 + (\beta t_{\mathbf{k}})^2]$$

$$F_1(\beta) = \prod_{\nu} [1 - R(\nu)]^{-1/2}, \qquad R(\nu) = \sum_{\mathbf{k}} R(\mathbf{k}, \nu) \frac{2U\beta}{N}$$

Let us transform the expression for the QPF to a form suitable for analysis. For this purpose one has to make the summation over the frequencies ν in expression (48) for $F_1(\beta)$. Now it is possible to represent the function $F_1(\beta)$

in the form

$$F_1(\beta)=\exp\left\{-\tfrac{1}{2}\sum_{\nu}\ln[1-R(\nu)]\right\} \tag{49}$$

Carrying out the expansion of the logarithm in the Taylor series [the expansion condition coincides exactly with condition (45) of the QPF convergence when $b_2=1$ and $a_{\mathbf{k}}=0$] and using relation (44), we obtain the following expression:

$$F_1(\beta)=\exp\left\{-\frac{1}{4N}\sum_{\mathbf{k}}\lambda^2(\mathbf{k})h_{\mathbf{k}}(\beta)\left[1-\tfrac{1}{2}ch^{-2}\left(\frac{\beta t_{\mathbf{k}}}{2}\right)\right]\right\} \tag{50}$$

$$h_{\mathbf{k}}(\beta)=\sum_{n=1}^{\infty}\frac{(-2U\beta)^n}{n}f_{n,\mathbf{k}}(\beta)$$

where the functions $f_{n,\mathbf{k}}(\beta)$ are defined by the recurrence relations

$$f_{n,\mathbf{k}}(\beta)=\frac{1}{N}\sum_{\mathbf{k}_1}\lambda^2(\mathbf{k}_1)[f_{n-1,\mathbf{k}}(\beta)+f_{n-1,\mathbf{k}_1}(\beta)]\frac{th(\beta t_{\mathbf{k}_1}/2)}{2\beta(t_{\mathbf{k}}-t_{\mathbf{k}_1})},\qquad f_{1,\mathbf{k}}(\beta)=1$$

The important consequence of the circumstance that our solution for the QPF was obtained exactly is in the fact that it contains as singularities the first-degree branch points, which cannot enter into the theory in which the summation of any Feynman diagram sequence of the geometric progression type is carried out.

The QPF diverges in the region where condition (38) of the inclusion of the value under consideration into the interval $(-1,1]$

$$\frac{2U\beta}{N}\sum_{\mathbf{k}}\lambda^2(\mathbf{k})\frac{th(\beta t_{\mathbf{k}}/2)}{i\hbar\beta\nu-2\beta t_{\mathbf{k}}}\in(-1,1] \tag{51}$$

is not fulfilled. This divergence is probably a consequence of the fact that the interaction Hamiltonian V_2 in (6) of our problem is singular, as it describes the equal interaction between the Cooper pairs, which does not depend on distance. It is clear from expression (48) for $F_1(\beta)$ that the reduced BCS model QPF can possess only a singularity on the real axis due to the central factor, corresponding to $\nu=0$. The existence condition of this singularity gives an equation which defines a critical temperature T_c^*:

$$\lim_{N\to\infty}\frac{U}{N}\sum_{\mathbf{k}}\lambda^2(\mathbf{k})\frac{th(\beta_c^* t_{\mathbf{k}}/2)}{t_{\mathbf{k}}}=1,\qquad \beta_c^*=\frac{1}{kT_c^*} \tag{52}$$

Solving this equation by traditional methods (Bogolyubov, 1972), we obtain the following expression for the temperature T_c^*:

$$T_c^*=T_{\mathrm{D}}\,e^{-1/2\lambda}=(T_{\mathrm{D}}T_c)^{1/2} \tag{53}$$

On the other hand, it follows from condition (51) that for the convergence of the QPF it is enough to demand the fulfillment of condition (51) only for $\nu = 0$. Hence, it is clear that the region of convergence of the reduced BCS model QPF in the nonregular phase is restricted from below by some critical temperature T_c^*, which is given by relation (53). Thus, the sense of this temperature is that it is an indication of some temperature boundary to which one can work with the interaction Hamiltonian V_2. Below T_c^* the fluctuations in the system sharply increase and due to the equality of the interactions between all Cooper pairs, given by the Hamiltonian V_2, they result in the divergence of the QPF.

For the regular phase (in the temperature limit $T < T_c$) in the thermodynamic limit the interaction Hamiltonian $V_1 + V_2$ is equivalent to the so-called approximation Hamiltonian, which has the same form as V_1 with a renormalized interaction constant (Bogolyubov, 1972). In this case the expression for the QPF coincides with result (46) and the critical temperature estimation with (47). The above definition of T_c^* has been obtained from the theory developed for the nonregular phase (in the temperature region $T > T_c^*$), when the four-fermion interaction Hamiltonian V_2 cannot turn into an approximation Hamiltonian of the V_1 type. Thus, if we treat the critical temperatures as the limit points up to which the exact solutions in the regular and nonregular phases exist, the estimations of their values do not coincide and the inequality $T_c^* > T_c$ always holds. In the intermediate temperature interval $T_c < T < T_c^*$ the fluctuations are so large that for the correct description of our system in this interval it seems necessary to reconstruct the model Hamiltonian.

From the methodological point of view developed for the QPF calculation by the path integration method, it is interesting to deduce the QPF expression in the GCS-2 basis, which is given by relation (13). In this basis the action operator covariant symbol has the form

$$\frac{A_c}{\hbar} = \lim_{N_T \to \infty} \sum_{\tau} \frac{A_c(\tau)}{\hbar} \tag{54}$$

$$\begin{aligned}\frac{A_c(\tau)}{\hbar} = \sum_{\mathbf{k}} \{ & \Psi_{c,1/2}^*(\mathbf{k}, \tau)\, W(\mathbf{k}, \tau) \Psi_{c,1/2}(\mathbf{k}, \tau) \\ & + \Psi_{c,-1/2}^*(-\mathbf{k}, \tau)\, W(-\mathbf{k}, \tau) \Psi_{c,-1/2}(-\mathbf{k}, \tau) \\ & + \Delta\beta\lambda(\mathbf{k})[\Delta_c(\mathbf{k}, \tau) + \Delta_c^*(\mathbf{k}, \tau)] \} \\ & + \frac{U\beta}{N} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \lambda(\mathbf{k}_1)\lambda(\mathbf{k}_2)\Delta_c^*(\mathbf{k}_1, \tau)\Delta_c(\mathbf{k}_2, \tau)\end{aligned}$$

Carrying out now the calculations according to the scheme described in Section 2, one can deduce the following expression for the QPF in the thermodynamic limit:

$$Z_2(\beta)=Z \lim_{N\to\infty} \prod_{\omega} \lim_{b_1\to 0} \lim_{b_2\to 0} G(b_1, b_2)$$

$$\times \prod_{\mathbf{k}} \lim_{a_{\mathbf{k}}\to 0} \{(\hbar\beta\omega)^2+(\beta t_{\mathbf{k}})^2+[\Delta\beta\lambda(\mathbf{k})]^2 e^{-d/da_{\mathbf{k}}}\}$$

$$\times\left[1-b_2\frac{U\beta}{N}\sum_{\mathbf{k}}(1+a_{\mathbf{k}})\mathrm{R}\left(\mathbf{k}, \omega-\frac{\pi}{\hbar\beta}\right)\right]^{-1} \tag{55}$$

As in the case of the QPF $Z_1(\beta)$ analysis, we discuss the $Z_2(\beta)$ properties in two limiting cases.

1. $U=0$. In this case the action operator covariant symbol (55) coincides with action operator covariant symbol (19), which is found in the GCS-1 basis. It leads to the same results (46) and (47) in the regular phase for the QPF and critical temperature estimation.

2. $\Delta=0$. In this case expression (55) transforms into the following result for the nonregular phase:

$$Z_2(\beta)|_{\Delta=0}=\mathrm{Sp}\, e^{-\beta(H_0+V_2)} \Rightarrow \lim_{N\to\infty} \mathrm{Sp}\, e^{-\beta H_0} F_2(\beta) \tag{56}$$

$$F_2(\beta)=\prod_{\nu}[1-R(\nu)/2]^{-1}$$

which coincides with the result for the QPF obtained earlier by the path integration method for the same model with the Hamiltonian H_0+V_2 (Popov, 1981). In the process of obtaining result (56) (Popov, 1981) the stationary phase approximation in the thermodynamic limit was used. From another point of view it is shown now that this solution (Popov, 1981) is the exact one in the thermodynamic limit if the GCS-2 basis (13) is used. Thus, in the path integration method developed for our problem the use of the reduced GCS-2 basis is apparently equivalent to the application of the stationary phase approximation in the thermodynamic limit. The QPF $Z_2(\beta)|_{\Delta=0}$ analysis gives that expression (56) converges in the temperature region $T>T_c$, where T_c is given by relation (47). Hence, it is clear that in this case the critical temperature estimations both for the regular and nonregular phases coincide. As to the approximation Hamiltonian method (Bogolyubov, 1972) developed in the thermodynamic limit for the regular phase, it gives results which exactly coincide with the results obtained by the path integration method for the regular phase in both the GCS-1 and GCS-2 bases.

Summing up all these facts, one can conclude that the calculation results obtained by the path integration method for the QPF (and consequently for the other thermodynamic characteristics) depend on the choice of the GCS basis. The character of the QPF divergence changes (the branch point is substituted by the pole) and the critical temperature estimation is moved considerably in the nonregular phase.

Thus, the choice of the GCS form becomes the principal question. From the theoretical point of view one ought to give preference to the GCS-1 basis, as just such coherent states satisfy all known demands, including the group-theoretic one (see Section 1).

4. SOME REMARKS ON THE NATURE OF THE ARBITRARINESS IN THE PATH INTEGRATION CAUSED BY THE GCS CHOICE

The function $F(\beta)$, which describes the deviation of the QPF of the model with interaction from the QPF of the model without interaction, is, in essence, the S-matrix with the imaginary time parameter $i\tau$, where $\tau \in [0, \hbar\beta]$. This circumstance allows us to investigate the differences in the QPF $Z_1(\beta)$ and $Z_2(\beta)$ from the S-matrix point of view. It is shown by Bogolyubov and Shirkov (1959) in the axiomatic approach to the quantum scattering theory that every term of the S-matrix expansion in the coupling constant (from the second one), contains an arbitrariness in the form of the integrals over the arbitrary quasilocal operators. In particular, just with the help of this arbitrariness it became possible to carry out the renormalization of the quantum electrodynamics by attaching quite definite, nontrivial expressions to these quasilocal operators (Bogolyubov and Shirkov, 1959).

Treating the function $F(\beta)$ as the S-matrix with imaginary argument, one can apply the well-known S-matrix expansion in the coupling constant:

$$F(\beta) = 1 + F^{(1)}(\beta) + F^{(2)}(\beta) + \cdots \tag{57}$$

$$F^{(1)}(\beta) = -\int_0^\beta d\tau \, \langle V_2(\tau) \rangle_0 \tag{58}$$

$$F^{(2)}(\beta) = \frac{1}{2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \, \langle T[V_2(\tau_1) V_2(\tau_2)] \rangle_0 + \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \, \Lambda_2(\tau_1, \tau_2) \delta(\tau_1 - \tau_2) \tag{59}$$

$$T[V_2(\tau_1) V_2(\tau_2)] = \theta(\tau_1 - \tau_2) V_2(\tau_1) V_2(\tau_2) + \theta(\tau_2 - \tau_1) V_2(\tau_2) V_2(\tau_1)$$

$$\Lambda_2(\tau_1, \tau_2) = R(\tau_1, \tau_2) \theta(\beta - \tau_1) \theta(\beta - \tau_2)$$

where $V_2(\tau)=e^{\tau H_0}V_2\,e^{-\tau H_0}$, $\langle\cdots\rangle_0=\mathrm{Sp}(e^{-\beta H_0}\cdots)/\mathrm{Sp}\,e^{-\beta H_0}$, and $\theta(x)=0$ if $x\le 0$, $\theta(x)=1$ if $x>0$. Here $\Lambda_2(\tau_1,\tau_2)\delta(\tau_1-\tau_2)$ is a quasilocal operator with an arbitrary function of two arguments $R(\tau_1,\tau_2)$. For the simplicity of the further analysis we concentrate our attention on the case when $\Delta=0$. One can obtain that

$$F^{(1)}(\beta)=-\frac{U\beta}{2N}\sum_{\mathbf{k}}\lambda^2(\mathbf{k})[n_{\mathbf{k}}^2+(1-n_{\mathbf{k}})^2] \tag{60}$$

$$\begin{aligned}F^{(2)}(\beta)=&\frac{1}{2}\left(\frac{U\beta}{2N}\right)^2\sum_{\mathbf{p}}\sum_{\mathbf{q}}[\lambda(\mathbf{p})\lambda(\mathbf{q})]^2[n_{\mathbf{p}}^2+(1-n_{\mathbf{p}})^2][n_{\mathbf{q}}^2+(1-n_{\mathbf{q}})^2]\\&+\frac{1}{2}\left(\frac{U\beta}{N}\right)^2\sum_{\mathbf{p}\neq\mathbf{q}}\sum[\lambda(\mathbf{p})\lambda(\mathbf{q})]^2\frac{n_{\mathbf{p}}^2(1-n_{\mathbf{q}})^2}{\beta(t_{\mathbf{q}}-t_{\mathbf{p}})}\\&+\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\infty}d\tau_2\,\Lambda_2(\tau_1,\tau_2)\delta(\tau_1-\tau_2)\end{aligned} \tag{61}$$

The analogous expansions in the coupling constant $U\beta$ of the functions $F_1(\beta)$ and $F_2(\beta)$ obtained by the path integration method with the use of the GCS-1 and GCS-2 bases accordingly leads as expected, to nonequal results, which differ from each other in the terms proportional to the $(U\beta)^2$. Attaching to the arbitrary function $R(\tau_1,\tau_2)$ the following meaning,

$$\begin{aligned}R_2(\tau_1,\tau_2)=&\left(\frac{U\beta}{2N}\right)^2\sum_{\mathbf{p}\neq\mathbf{q}}\sum\frac{[\lambda(\mathbf{p})\lambda(\mathbf{q})]^2}{\beta(t_{\mathbf{q}}-t_{\mathbf{p}})}\\&\times\frac{e^{2\tau_1 t_{\mathbf{q}}}[t_{\mathbf{q}}(e^{\tau_2 t_{\mathbf{p}}}+1)+t_{\mathbf{p}}\,e^{\tau_2 t_{\mathbf{p}}}(e^{\tau_1 t_{\mathbf{q}}}+1)]}{(e^{\tau_1 t_{\mathbf{q}}}+1)^3(e^{\tau_2 t_{\mathbf{p}}}+1)^3}\end{aligned} \tag{62}$$

one can restore, with the help of formula (61), the expansion of the function $F_1(\beta)$ with an accuracy up to the $(U\beta)^2$ terms. For the analogous restoration of the function $F_2(\beta)$ it is enough to set down in expression (61) that $R_2(\tau_1,\tau_2)=0$. It should be noted (Izmailov and Kessel, 1989*a*–*c*) this fact was not indicated in the process of agreement of the expressions for QPF $Z_1(\beta)$ and $Z_2(\beta)$ with the perturbation theory. In these papers we took the S-matrix expansion within the axiomatic theory as the stationary perturbation theory with $\theta(x)=0$ if $x<0$, $1/2$ if $x=0$, and 1 if $x>0$, and $R_2(\tau_1,\tau_2)=0$. As was shown above, the application of the θ function defined by expression (59) leads to the same conclusions if the perturbation theory with the quasilocal operator in the form (59) and (62) is used.

Thus, these discussions reveal that the arbitrariness in the path integral calculations connected with the use of the different GCS bases apparently does not exceed the bounds of the arbitrariness which exists due to the arbitrary quasilocal operators in the S-matrix axiomatic theory. The appearance of the arbitrary quasilocal operators in the theory is deeply rooted:

the Schrödinger quantum mechanical picture is based on the solution of differential equations and so contains the arbitrariness in the integration constants (Bogolyubov and Shirkov, 1959).

ACKNOWLEDGMENTS

The authors are indebted to V. Ya. Fainberg for constant support and stimulating discussions and to D. A. Kirzhnits, E. S. Fradkin, and V. N. Popov for discussions and critical remarks.

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